BOUNDARY VALUE PROBLEMS
OF THE THEORY OF SHALLOW SHELLS

PMM Vol. 34, №2, 1970, pp. 258-269
L. A. FIL'SHTINSKII
(Novosibirsk)
(Received November 17, 1969)

The boundary value problems of shallow shells theory can provisionally be separated into two kinds, internal (analysis of domes, beamless ceilings), and external boundary value problems (analysis of shells weakened by holes, i.e. problems of stress concentration). Both these directions have extensive bibliographies (see [1], say).

Given below is a formulation of the fundamental boundary value problems from rather unique positions, similar in idea to the Koslov-Muskhelishvili conceptions in the plane problem of elasticity theory. Representations of the solutions of the boundary value problems are written down in series for simply and multiply connected domains.

1. The correctly written representations of solutions of boundary value problems in shell theory should satisfy the following conditions: (a) functions governing the displacements, stress resultants, and moments in the shell should be expressed in terms of the general solutions of the fundamental equations or be represented as series in the complete systems of solutions, (b) the static conditions as a whole should be satisfied, i.e. conditions at infinity (if the shell is assumed unbounded or the equilibrium conditions of the whole shell if the shell is bounded), (c) conditions of uniqueness of the displacements (if there are no dislocations according to the meaning of the problem, (d) finally, in some problems it is necessary to impose conditions of periodicity of the displacements or stress resultants. For example, in the tension of a cylindrical shell weakened by a large quantity of periodically disposed identical holes, the stress resultants should satisfy the appropriate periodicity conditions.

The general solution of the fundamental equations of technical shallow shell theory is given in [2]; we shall consider it known. The stress and deflection functions are defined in terms of the general solution $F(z, \zeta)$ as follows:

$$U(x, y) = F_1(z, \zeta), \quad w(x, y) = \varepsilon^* F_2(z, \zeta), \quad F(z, \zeta) = F_1 + iF_2$$

$$z = \frac{\beta\sqrt{i}}{a}(x + iy), \qquad \zeta = \frac{\beta\sqrt{i}}{a}(x - iy), \qquad \beta = \frac{\sqrt{\varepsilon(1-\alpha)}}{4} \tag{1.1}$$

$$\varepsilon = \frac{a^2\sqrt{12(1-\mu^2)}}{Rh}$$

$$\varepsilon^* = \frac{\sqrt{12(1-\mu^2)}}{Eh^2}, \qquad \alpha = \frac{R}{R_1}, \qquad |\alpha| \leqslant 1$$

Here $U(x, y)$ and $w(x, y)$ are the stress and deflection functions, E, μ and h the Young's modulus, Poisson's ratio, and shell thickness, respectively, R, R_1 are the corresponding radii of curvature of the middle surface, x, y are Cartesian coordinates, and a is the characteristic linear dimension.

All the stress resultant and moments acting in the shell can be expressed in terms of the function $F(z, \zeta)$. To do this, it is merely necessary to go over to the variables z, ζ in the known formulas. We have

$$\frac{hN_x}{\sqrt{12(1-\mu^2)}} + i\frac{M_y - \mu M_x}{1-\mu^2} = \frac{i(1-\alpha)}{16R}\left(-\frac{\partial^2 F}{\partial z^2} + 2\frac{\partial^2 F}{\partial z\,\partial\zeta} - \frac{\partial^2 F}{\partial\zeta^2}\right)$$

$$\frac{hN_y}{\sqrt{12(1-\mu^2)}} + i\frac{M_x - \mu M_y}{1-\mu^2} = \frac{i(1-\alpha)}{16R}\left(\frac{\partial^2 F}{\partial z^2} + 2\frac{\partial^2 F}{\partial z\,\partial\zeta} + \frac{\partial^2 F}{\partial\zeta^2}\right) \quad (1.2)$$

$$\frac{hN_{xy}}{\sqrt{12(1-\mu^2)}} - i\frac{M_{xy}}{1-\mu} = \frac{1-\alpha}{16R}\left(\frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial\zeta^2}\right)$$

where N_x, N_y and N_{xy} are the stress resultants in the middle surface, M_x, M_y and M_{xy} are the corresponding moments in the shell.

Analogous formulas can obviously be written for the transverse stress resultants also.

Let us turn to the determination of the tangential displacements u and v. From Hooke's law connecting the stresses and strains in the shell we have

$$N_y - N_x + 2iN_{xy} = -4Gh\frac{\beta\sqrt{i}}{a}\frac{\partial}{\partial z}(u - iv) - 2Gh\varepsilon^*\frac{1-\alpha}{R}F_2 =$$
$$= 4\left(\frac{\beta\sqrt{i}}{a}\right)^2\frac{\partial^2 F_1}{\partial z^2} \quad (1.3)$$

Integrating (1.3) with respect to z we obtain

$$4Gh(u - iv) = \frac{a}{\beta\sqrt{i}}f(\zeta) - 4\frac{\beta\sqrt{i}}{a}\frac{\partial F_1}{\partial z} - 2Gh\frac{1-\alpha}{R}\frac{a}{\beta\sqrt{i}}\int_{z_0}^{z} w\,dz \quad (1.4)$$

The question therefore rests upon the determination of the analytic function $f(\zeta)$. Applying a procedure [3], we obtain

$$(1+\mu)a^2\,\mathrm{Re}\,f'(\zeta) = \frac{i(1-\alpha)\varepsilon}{2}\frac{\partial^2 F_1}{\partial z\,\partial\zeta} + (1+\alpha)\varepsilon F_2(z,\zeta) +$$
$$+ (1-\alpha)\varepsilon\,\mathrm{Re}\left\{\frac{\partial}{\partial\zeta}\int_{z_0}^{z} F_2\,dz\right\} \quad (1.5)$$

It is more convenient to rewrite this latter expression as

$$(1+\mu)a^2\,\mathrm{Re}\,f'(\zeta) = \frac{1-\alpha}{2}\varepsilon\,\mathrm{Im}\left\{-\frac{\partial^2 F}{\partial z\,\partial\zeta} + 2\delta F + \frac{\partial}{\partial\zeta}\int_{z_0}^{z} F\,dz + \frac{\partial}{\partial z}\int_{\zeta_0}^{\zeta} F\,d\zeta\right\} \quad (1.6)$$

Just one function $F(z, \zeta)$ figures in the right side of (1.6).

It is now necessary to take account of the fact that $F(z, \zeta)$ is a solution of the fundamental equation. According to [2], the following representations hold

$$F(z,\zeta) = \varphi_0(z)\,\mathrm{ch}(\zeta-\zeta_0) + \psi_0(\zeta)\,\mathrm{ch}(z-z_0) - \quad (1.7)$$
$$-\sum_{k=0}^{1}\int_{z_0}^{z}\varphi_k(t)\frac{\partial}{\partial t}G_k(z-t,\zeta-\zeta_0)\,dt - \sum_{k=0}^{1}\int_{\zeta_0}^{\zeta}\psi_k(\tau)\frac{\partial}{\partial\tau}G_k(z-z_0,\zeta-\tau)\,d\tau$$

Here $\varphi_k(z)$ and $\psi_k(\zeta)$ are arbitrary analytic functions of their arguments, and the kernels $G_k(z-t, \zeta-\tau)$ are known function.

Substituting (1.7) into the right side of (1.6), we obtain after transformations

$$(1+\mu)a^2\,\mathrm{Re}\,f'(\zeta) = -\frac{1-\alpha}{2}\varepsilon\,\mathrm{Im}\,[\varphi_1(z) + \psi_1(\zeta) \quad (1.8)$$

Restoring $f'(\zeta)$ and then integrating it, we find

$$\frac{Rhi\sqrt{1\pm\mu}}{(1-\alpha)\sqrt{3(1-\mu)}}f(\zeta)=-\int_{\zeta_0}^{\zeta}\psi_1(\zeta)\,d\zeta+\int_{\zeta_0}^{\zeta}\overline{\varphi}_1(-i\zeta)\,d\zeta \tag{1.9}$$

The function $\overline{\varphi}(z)$ in (1.9) is defined by the relationship $\overline{\varphi}(z)=\overline{\varphi(\bar{z})}$ [4].

By virtue of (1.9) and (1.4) the final expression of the tangential displacements can be represented as

$$\lambda(u-iv)=\sqrt{i}\left\{\int_{\zeta_0}^{\zeta}[\psi_1(\zeta)-\overline{\varphi}_1(-i\zeta)]\,d\zeta-\frac{1+\mu}{2}\frac{\partial}{\partial z}F_1+2i\int_{z_0}^{z}F_2\,dz\right\} \tag{1.10}$$

where

$$\lambda=Eh\left\{\frac{Rh}{(1-\alpha)\sqrt{12(1-\mu^2)}}\right\}^{1/2},\qquad F_1=\operatorname{Re}F(z,\zeta),\qquad F_2=\operatorname{Im}F(z,\zeta)$$

The function $F(z,\ \zeta)$ in (1.10) is given by the representation (1.7); the functions $\psi_1\ (\zeta)$ and $\varphi_1(z)$ are arbitrary analytic functions in (1.7). Let us examine the static conditions. The principal stress resultant vector in the middle surface acting along the arbitrary arc L on the shell surface is defined by the formula

$$X-iY=-2i\frac{\beta\sqrt{i}}{a}\frac{\partial F_1}{\partial z}\Big|_L \tag{1.11}$$

Furthermore, the projection of the principal vector of the forces along L on the normal is

$$Q=\int_L\left\{\frac{y}{R}d\left(\frac{\partial U}{\partial x}\right)-\frac{x}{R_1}d\left(\frac{\partial U}{\partial y}\right)+D\frac{\partial}{\partial x}\nabla^2w\,dy-D\frac{\partial}{\partial y}\nabla^2w\,dx\right\} \tag{1.12}$$

Here $D=Eh^3\ /\ 12\ (1\ -\mu^2)$ is the cylindrical stiffness. This latter expression can be represented in the more conveneient form

$$Q=\frac{1-\alpha}{4R}\operatorname{Im}\left\{2(\delta z-\zeta)\frac{\partial F}{\partial z}\Big|_L-2(\delta\zeta-z)\frac{\partial F}{\partial\zeta}\Big|_L+\right. \tag{1.13}$$

$$\left.+\int_L\left(\frac{\partial^3F}{\partial z^2\,\partial\zeta}-2\frac{\partial F}{\partial\zeta}-2\delta\frac{\partial F}{\partial z}\right)z'(s)\,ds-\int_L\left(\frac{\partial^3F}{\partial z\,\partial\zeta^2}-2\frac{\partial F}{\partial z}-2\delta\frac{\partial F}{\partial\zeta}\right)\zeta'(s)\,ds\right\}$$

2. In the case of the first boundary value problem we shall assume that the stress resultants and moments

$$N_n=N_n^\circ(s),\quad N_s=N_s^\circ(s),\quad M_n=M_n^\circ(s),\quad Q^*=Q_n+\frac{\partial M_{ns}}{\partial s}=Q^\circ(s) \tag{2.1}$$

are given on the domain boundary L, where N_n and N_s are the normal and shear components of the stress resultant in the middle surface, M_n and Q^* are the normal component of the bending moment and the generalized, in the Kirchhoff sense, transverse stress resultant, and s is the arc coordinate along the boundary.

The boundary conditions (2.1) can be represented in the following equivalent form:

$$N_n-iN_s=f_1^\circ(s),\qquad M_n-i\left(M_s+\int_0^s Q_n\,ds+\text{const}\right)=f_2^\circ(s) \tag{2.2}$$

where

$$f_1^\circ(s)=N_n^\circ-iN_s^\circ,\qquad f_2^\circ(s)=M_n^\circ-i\int_0^s Q^\circ(s)\,ds+\text{const}$$

Let

$$F_s(z,\zeta)=F(z,\zeta)+F^\circ(z,\zeta) \tag{2.3}$$

where $F^\circ\ (z,\ \zeta)$ is some particular solution of the inhomogeneous equation (which corresponds either to some loading at infinity if the shell is unbounded, or to a transverse loading on the shell), F is the solution of the fundamental homogeneous equation of shallow shell theory [2].

The boundary conditions (2.2) can be expressed in terms of the boundary values of the function F_s as follows:

$$\frac{\partial^2 F_s}{\partial t\,\partial\tau} - \left(\overline{\frac{\partial^2 F_s}{\partial t\,\partial\tau}}\right) - e^{2i\theta}\left\{\frac{\partial^2 F_s}{\partial t^2} - \left(\overline{\frac{\partial^2 F_s}{\partial\tau^2}}\right)\right\} = 2f_1(s)$$

$$(1+\mu)\left[\frac{\partial^2 F_s}{\partial t\,\partial\tau} + \left(\overline{\frac{\partial^2 F_s}{\partial t\,\partial\tau}}\right)\right] + (1-\mu)e^{2i\theta}\left[\frac{\partial^2 F_s}{\partial t^2} + \left(\overline{\frac{\partial^2 F_s}{\partial\tau^2}}\right)\right] +$$

$$+ 4i\,\mathrm{Im}\int_0^s\left[t'(s)\frac{\partial^3 F_s}{\partial t^2\,\partial\tau} - \tau'(s)\frac{\partial^3 F_s}{\partial\tau^2\,\partial t}\right]ds = 2if_2(s) \qquad (2.4)$$

where

$$f_1(s) = \frac{a^2}{2i\beta^2}f_1^\circ(s), \quad f_2(s) = \frac{4R}{i(1-\alpha)}f_2^\circ(s), \quad t(s) = \frac{\beta\sqrt{i}}{a}[x(s)+iy(s)]$$

$$\tau(s) = \frac{\beta\sqrt{i}}{a}[x(s) - iy(s)]$$

The functions $f_1^\circ(s)$ and $f_2^\circ(s)$ are given in (2.2).

In the case of the second fundamental problem, we give the tangential displacements u, v, the deflection w, and the normal derivative on L. The boundary conditions can evidently be represented as follows:

$$u - iv = u^\circ(s) - iv^\circ(s), \qquad \frac{\partial w}{\partial s} - i\frac{\partial w}{\partial n} = \frac{\partial w^\circ(s)}{\partial s} - iw_n^\circ(s) \qquad (2.5)$$

Utilizing the formulas for the tangential displacements (1.10), we reduce condition (2.5) to

$$\int_{\tau_0(s)}^{\tau(s)}[\psi_1(\zeta) - \overline{\varphi}_1(-i\zeta)]\,d\zeta - \frac{1+\mu}{2}\frac{\partial F_1}{\partial t} + 2i\int_{t_0(s)}^{t(s)}F_2 t'(s)\,ds = f_3(s) \quad (2.6)$$

$$\frac{\partial F_2}{\partial t} = e^{i\theta}f_4(s), \quad f_3(s) = \frac{\lambda}{\sqrt{i}}(u^\circ - iv^\circ), \quad f_4(s) = \frac{a\sqrt{i}}{2\beta\varepsilon^*}\left(\frac{\partial w^\circ}{\partial s} - iw_n^\circ\right)$$

Here n is the direction of the exterior normal to L.

Several analogous boundary value problems exist besides the two mentioned.

The question therefore reduces to seeking the solution of the fundamental equation $F(z, \zeta)$ from certain boundary conditions on the domain boundary. The representation $F(z, \zeta)$ should hence satisfy the correctness conditions.

3. If the domain is simply connected and finite, the solution of boundary value problems 1 and 2 can be represented as series in some complete system of partial solutions. For example (3.1)

$$F(z, \zeta) = \sum_{n=0}^{\infty}\{A_n\Phi_n(z, \zeta) + A_n^*\Phi_n^*(z, \zeta) + B_n\Psi_n(z, \zeta) + B_n^*\Psi_n^*(z, \zeta)\}$$

Here A_n, A_n^*, B_n and B_n^* are constants to be determined from the boundary conditions; the functions Φ_n, Φ_n^*, Ψ_n and Ψ_n^* have been constructed in [2].

If the domain is circular, we use a polar coordinate representation of the solution [2]; if the domain is not a circle, one of the approximate methods, say the method of boundary collocation, can be utilized upon compliance with the boundary conditions.

4. In solving the boundary value problem for an infinite domain it is necessary to have the solution which damps at infinity. A construction of such solutions is given below.

Let us introduce the function

$$G(z,\zeta)=G_0(z,\zeta)+\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \zeta}\right)G_1(z,\zeta)$$

where G_0 and G_1 are the kernels in (1.7). We represent the general solution $F(z,\zeta)$ as

$$F(z,\zeta)=a_0G(z-z_0,\zeta-\zeta_0)+a_1G(z_0-z,\zeta_0-\zeta)+\int_{z_0}^{z}G(z-t,\zeta-\zeta_0)\times$$
$$\times\mu_0(t)\,dt+\int_{z}^{z_0}G(t-z,\zeta_0-\zeta)\mu_1(-t)\,dt+\int_{\zeta_0}^{\zeta}G(z-z_0,\zeta-\tau)\nu_0(\tau)\,d\tau+$$
$$+\int_{\zeta}^{\zeta_0}G(z_0-z,\tau-\zeta)\nu_1(-\tau)\,d\tau \tag{4.2}$$

Here μ_0, μ_1, ν_0 and ν_1 are arbitrary analytic functions of their arguments, a_0 and a_1 are arbitrary constants.

The representation (4.2) reflects the following property of the solution of the fundamental equation. If $F(z,\zeta)$ is a solution, the functions $F(\zeta, z)$, $F(-z,-\zeta)$, $F(-\zeta,-z)$ are also solutions.

Let us introduce four kinds of solutions

$$\Phi(z,\zeta)=$$
$$=D^{\circ}_{z,\zeta}\{\mu_0(z-z_0)\}=\frac{a_0}{2}G(z-z_0,\zeta-\zeta_0)+\int_{z_0}^{z}G(z-t,\zeta-\zeta_0)\mu_0(t-z_0)\,dt$$
$$\Psi(z,\zeta)=$$
$$=D'_{z,\zeta}\{\mu_1(z_0-z)\}=\frac{a_1}{2}G(z_0-z,\zeta_0-\zeta)+\int_{z}^{z_0}G(t-z,\zeta_0-\zeta)\mu_1(z_0-t)\,dt$$
$$\Phi^*(z,\zeta)=$$
$$=D^{\circ}_{\zeta,z}\{\nu_0(\zeta-\zeta_0)\}=\frac{a_0}{2}G(z-z_0,\zeta-\zeta_0)+\int_{\zeta_0}^{\zeta}G(z-z_0,\zeta-\tau)\nu_0(\tau-\zeta_0)\,d\tau$$
$$\Psi^*(z,\zeta)= \tag{4.3}$$
$$=D^{1}_{\zeta,z}\{\nu_1(\zeta_0-\zeta)\}=\frac{a_1}{2}G(z_0-z,\zeta_0-\zeta)+\int_{\zeta}^{\zeta_0}G(z_0-z,\tau-\zeta)\nu_1(\zeta_0-\tau)\,d\tau$$

The general solution of the fundamental equation of shallow shell theory is evidently

$$F(z,\zeta)=\Phi(z,\zeta)+\Psi(z,\zeta)+\Phi^*(z,\zeta)+\Psi^*(z,\zeta) \tag{4.4}$$

Let us consider particular solutions which we shall call solutions of the first kind

$$\Phi_\gamma(z-z_0,\zeta-\zeta_0)=$$
$$=D^{\circ}_{z,\zeta}\left\{\frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)}e^{z-z_0}\right\},\quad \Psi_\gamma(z-z_0,\zeta-\zeta_0)=D^{1}_{z,\zeta}\left\{\frac{(z_0-z)^{\gamma-1}}{\Gamma(\gamma)}e^{z_0-z}\right\} \tag{4.5}$$

$$\Phi_\gamma^*(z-z_0, \zeta-\zeta_0) = \qquad \text{(cont.)}$$

$$= D^{\circ}_{\zeta, z}\left\{\frac{(\zeta-\zeta_0)^{\gamma-1}}{\Gamma(\gamma)} e^{\zeta-\zeta_0}\right\}, \quad \Psi_\gamma^*(z-z_0, \zeta-\zeta_0) = D'_{\zeta, z}\left\{\frac{(\zeta_0-\zeta)^{\gamma-1}}{\Gamma(\gamma)} e^{\zeta_0-\zeta}\right\}$$

Here $\Gamma(\gamma)$ is the Euler gamma-function; we shall here assume that $\operatorname{Re}\gamma > 0$.

According to [2], the function $G(z, \zeta)$ can be represented as

$$G(z, \zeta) = \sum_{k=0}^{\infty} \frac{z^k}{k!} g_k'(\zeta) = e^{z+\zeta} \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k(\zeta) = G(\zeta, z) \tag{4.6}$$

where g_k and ω_k are lnown functions, for example, for a cylindrical shell ($\delta = 1$) we have

$$\omega_k(\zeta) = \frac{\zeta^k}{k!} \tag{4.7}$$

The equalities

$$\Psi_\gamma(z, \zeta) = \Phi_\gamma(-z, -\zeta), \quad \Phi_\gamma^*(z, \zeta) = \Phi_\gamma(\zeta, z),$$
$$\Psi_\gamma^*(z, \zeta) = \Phi_\gamma(-\zeta, -z) \tag{4.8}$$

follow from (4.5).

Therefore, only the function $\Phi_\gamma(z, \zeta)$ needs to be determined.

Realizing the first of formulas (4.5), taking account of (4.3) and (4.6), we find (*)

$$\Phi_\gamma(z, \zeta) = e^{z+\zeta} \sum_{k=0}^{\infty} \frac{z^{k+\gamma}\omega_k(\zeta)}{\Gamma(k+\gamma+1)} \tag{4.9}$$

For $\gamma = -n$, $(n = 1, 2\ldots)$, formula (4.9) becomes

$$\Phi_{-n}(z, \zeta) = e^{z+\zeta} \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_{k+n}(\zeta) \tag{4.10}$$

Formulas (4.9) and (4.10) yield an analytic continuation of integrals (4.5) into the whole plane of the parameter γ.

For the particular case of $\delta = 1$ (cylindrical shell), the system of regular solutions of the first kind yields

$$\Phi_\gamma(z, \zeta) = \left(\frac{z}{\zeta}\right)^{\gamma/2} e^{z+\zeta} I_\gamma(2\sqrt{z\zeta}), \quad \Psi_\gamma(z, \zeta) = \left(\frac{z}{\zeta}\right)^{\gamma/2} e^{-z-\zeta} I_\gamma(2\sqrt{z\zeta}) \tag{4.11}$$
$$\Phi_\gamma^*(z, \zeta) = \left(\frac{\zeta}{z}\right)^{\gamma/2} e^{z+\zeta} I_\gamma(2\sqrt{z\zeta}), \quad \Psi_\gamma^*(z, \zeta) = \left(\frac{\zeta}{z}\right)^{\gamma/2} e^{-z-\zeta} I_\gamma(2\sqrt{z\zeta})$$

Starting from the solutions (4.9), (4.8) obtained above, we construct solutions which decrease at infinity.

Let us define single-valued solutions of logarithmic type in terms of regular solutions of the first kind as follows:

$$\Omega_1(z, \zeta) = \frac{1}{2} \lim_{\gamma\to 0} \left\{\frac{\partial \Phi_{-\gamma}(z, \zeta)}{\partial\gamma} - \frac{\partial \Phi_\gamma^*(z, \zeta)}{\partial\gamma}\right\}$$
$$\Omega_2(z, \zeta) = \frac{1}{2} \lim_{\gamma\to 0} \left\{\frac{\partial \Psi_{-\gamma}(z, \zeta)}{\partial\gamma} - \frac{\partial \Psi_\gamma^*(z, \zeta)}{\partial\gamma}\right\} \tag{4.12}$$

*) We set $z_0 = \zeta_0 = 0$ everywhere below.

Realizing the first equality in (4.12), we find after an appropriate passage to the limit

$$\Omega_1(z, \zeta) = -G(z, \zeta)\ln\sqrt{z\zeta} + \Omega_1{}^\circ(z, \zeta) \tag{4.13}$$

$$\Omega_1{}^\circ(z, \zeta) = e^{z+\zeta}\sum_{k=0}^{\infty}\frac{\psi(k+1)}{2k!}[z^k\omega_k(\zeta) + \zeta^k\omega_k(z)], \quad \psi(k+1) = -C + \sum_{j=1}^{k}\frac{1}{j}$$

$\psi(1) = -C$ (C is the Euler constant).

It follows from (4.13) that $\Omega_1(z, \zeta)$ has a logarithmic singularity at the point $z = \zeta = 0$. The factor in the logarithm is the kernel, and the function $\Omega_1{}^\circ(z, \zeta)$ is analytic at any finite point z, ζ.

The second logarithmic solution is defined by virtue of (4.12) and (4.8) as

$$\Omega_2(z, \zeta) = \Omega_1(-z, -\zeta) \tag{4.14}$$

Let us designate as regular solutions of the second kind functions expressed as follows in terms of solutions of logarithmic type:

$$Z_1^{(-n)}(z, \zeta) = (-1)^n\left(\frac{\partial}{\partial z} - 1\right)^n\Omega_1(z, \zeta), \quad Z_2^{(-n)}(z, \zeta) = (-1)^n\left(\frac{\partial}{\partial \zeta} - 1\right)^n\Omega_1(z,\zeta)$$

$$Z_3^{(-n)}(z, \zeta) = Z_1^{(-n)}(-z, -\zeta), \qquad Z_4^{(-n)}(z, \zeta) = Z_2^{(-n)}(-z, -\zeta) \tag{4.15}$$

It is clear that solutions thus defined are single-valued and have a pole of order n at the point $z = \zeta = 0$.

From (4.15) and (4.13) we find

$$Z_1^{(-n)}(z, \zeta) = (-1)^{n+1}\Phi_{-n}(z, \zeta)\ln\sqrt{z\zeta} + e^{z+\zeta}\sum_{k=1}^{n}\frac{(-1)^{n-k}(k-1)!}{2z^k}\omega_{n-k}(\zeta) +$$

$$+ (-1)^n e^{z+\zeta}\sum_{k=0}^{\infty}\frac{\psi(k+1)}{2k!}\left[z^k\omega_{k+n}(\zeta) + \zeta^k\frac{d^n\omega_k(z)}{dz^n}\right] \tag{4.16}$$

The following equality holds

$$Z_2^{(-n)}(z, \zeta) = Z_1^{(-n)}(\zeta, z) \tag{4.17}$$

The remaining solutions of the second kind are defined in (4.15).

For the cylindrical shell the solutions (4.15) become

$$Z_1^{(-n)}(z, \zeta) = \left(\frac{\zeta}{z}\right)^{n/2} e^{z+\zeta}K_n(2\sqrt{z\zeta}), \quad Z_2^{(-n)}(z, \zeta) = \left(\frac{z}{\zeta}\right)^{n/2} e^{z+\zeta}K_n(2\sqrt{z\zeta})$$

$$Z_3^{(-n)}(z, \zeta) = \left(\frac{\zeta}{z}\right)^{n/2} e^{-z-\zeta}K_n(2\sqrt{z\zeta}), \quad Z_4^{(-n)}(z, \zeta) = \left(\frac{z}{\zeta}\right)^{n/2} e^{-z-\zeta}K_n(2\sqrt{z\zeta}) \tag{4.18}$$

where $K_n(t)$ is the Macdonald cylinder function.

In this case the logarithmic solutions (4.12) are expressed by the formulas

$$\Omega_1(z, \zeta) = e^{z+\zeta}K_0(2\sqrt{z\zeta}), \qquad \Omega_2(z, \zeta) = e^{-z-\zeta}K_0(2\sqrt{z\zeta}) \tag{4.19}$$

The order of decrease of the solutions (4.18), (4.19) as $\rho = \sqrt{x^2+y^2}$ increases is found by utilizing the asymptotic formulas for $K_\nu(x)$ for large $|x| \gg |\nu|$ (see [5]). We have

$$Z_i^{(-n)}(z, \zeta) \approx \left(\frac{\pi\sqrt{Rh}}{\rho\lambda}\right)^{1/2}\exp\left\{\frac{\lambda(x-\rho)}{2\sqrt{2Rh}}\right\} \quad (i = 0, 2) \tag{4.20}$$

$$Z_i^{(-n)}(z, \zeta) \approx \left(\frac{\pi\sqrt{Rh}}{\rho\lambda}\right)^{1/2}\exp\left\{\frac{\lambda(-x-\rho)}{2\sqrt{2Rh}}\right\} \quad (i = 1, 3) \quad \lambda = [12(1-\mu^2)]^{1/4}$$

As should have been expected [6], the least decay holds along the asymptotic line ($y = 0$).

Solutions, which decay at infinity and are defined in terms of the logarithmic solutions as follows:

$$T_1^{(-n)}(z, \zeta) = \frac{\partial^n}{\partial z^n}\Omega_1(z, \zeta), \qquad T_2^{(-n)}(z, \zeta) = T_1^{(-n)}(\zeta, z) \tag{4.21}$$

$$T_3^{(-n)}(z, \zeta) = T_1^{(-n)}(-z, -\zeta), \qquad T_4^{(-n)}(z, \zeta) = T_1^{(-n)}(-\zeta, -z)$$

are useful in solving the boundary value problems for an unbounded domain.

Let us use the notation

$$u_{p,q}^{j,n} = \frac{\partial^{p+q}}{\partial z^p \partial \zeta^q} T_j^{(-n)}(z, \zeta) \qquad (j = 1, 2, 3, 4) \tag{4.22}$$

Let us find the representation of the function $u_{p,q}^{j,n}$ in polar coordinates. To do this we write

$$u_{p,q}^{j,n} = \sum_{k=-\infty}^{\infty} F_{p,q}^{k,n}(j, \sqrt{z\zeta})\, e^{ik\theta}, \qquad e^{i\theta} = \left(\frac{z}{\zeta}\right)^{1/2} \tag{4.23}$$

Since by virtue of (4.21)

$$F_{p,q}^{k,n}(2, \sqrt{z\zeta}) = F_{q,p}^{-k,n}(1, \sqrt{z\zeta}), \quad F_{p,q}^{k,n}(3, \sqrt{z\zeta}) = (-1)^{k+p+q} F_{p,q}^{k,n}(1, \sqrt{z\zeta})$$

$$F_{p,q}^{k,n}(4, \sqrt{z\zeta}) = (-1)^{k+p+q} F_{q,p}^{-k,n}(1, \sqrt{z\zeta}) \tag{4.24}$$

then only the functions

$$F_{p,q}^{k,n}(1, \sqrt{z\zeta}) = F_{p,q}^{k,n}(\sqrt{z\zeta})$$

are needed in the definition.

The logarithmic solution $\Omega_1(z, \zeta)$ (4.13) can also be represented as follows:

$$\Omega_1(z, \zeta) = -G(z, \zeta)\ln\sqrt{z\zeta} + \Omega^\circ_1(z, \zeta) \tag{4.25}$$

$$\Omega_1^\circ(z, \zeta) = \sum_{k=0}^{\infty} \frac{\psi(k+1)}{2k!}[z^k \omega_k'(\zeta) + \zeta^k \omega_k'(z)] -$$

$$-\frac{1}{2}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!}\omega_k''(\zeta) - \frac{1}{2}\sum_{k=0}^{\infty}\frac{\zeta^{k+1}}{(k+1)!}\omega_k''(z)$$

$$\omega_k'(\zeta) = \sum_{s=0}^{\infty}\frac{\zeta^s}{s!}c_{k,s}, \qquad \omega_k''(\zeta) = \sum_{s=0}^{\infty}\frac{\zeta^s}{s!}c_k^{*s}, \qquad c_{k,s}^{*} = \sum_{j=0}^{k}\frac{c_{j,s}}{k-j+1}$$

$$c_{2k,2s} = a_{k,s}, \; c_{2k,2s+1} = a_{k,s} + b_{k-1,s}, \quad c_{2k+1,2s} = a_{k,s} + b_{k,s-1}, \quad c_{2k+1,2s+1} = b_{k,s}$$

$$b_{-1,s} = b_{k,-1} = 0 \qquad (s, k = 0, 1, \ldots)$$

The quantities $a_{k,s}$ and $b_{k,s}$ are defined in [2].

Differentiating (4.25) according to (4.21) and (4.22) and then representing the expression obtained in polar coordinates taking account of the easily deducible formula

$$\sum_{k=0}^{n}\frac{A_k}{z^{k+1}}\sum_{s=0}^{\infty} c_{n-k,s}\frac{\zeta^s}{s!} = \sum_{k=1}^{\infty} e^{-ik\theta}\sum_{s=0}^{\min(n,k-1)}\frac{A_s c_{n-s,k-s-1}}{(k-s-1)!}(\sqrt{z\zeta})^{k-2s-2} \tag{4.26}$$

we obtain

$$F_{p,q}^{k,n}(\sqrt{z\zeta}) = F_{0,q}^{k,n+p}(\sqrt{z\zeta}) \tag{4.27}$$

Here

$$F_{0,q}^{k,n}(\sqrt{z\zeta}) = \sum_{j=0}^{\min(q-1,\,k-1)} \frac{(-1)^j j!\, c_{q-j-1,\,k-j-1+n}}{2\Gamma(k-j)} (\sqrt{z\zeta})^{k-2j-2} -$$

$$-\sum_{j=0}^{\infty} \frac{(\sqrt{z\zeta})^{2j+k}}{j!\Gamma(j+k+1)} \left\{ c_{j+n+k,\, j+q} \left(\ln \sqrt{z\zeta} - \frac{\psi(j+1)+\psi(j+k+1)}{2} \right) + \right.$$

$$\left. + \frac{1}{2} (c^{*}_{j+k+n-1,\, j+q} + c^{*}_{j+q-1,\, j+k+n}) \right\}$$

$$(k = 0, 1, 2, \ldots)$$

$$F_{0,q}^{-k,n}(\sqrt{z\zeta}) = \sum_{j=0}^{\min(n-1,\,k-1)} \frac{(-1)^{j+1} j!\, c_{n-j-1,\, k-j+q-1}}{2\Gamma(k-j)} (\sqrt{z\zeta})^{k-2j-2} -$$

$$-\sum_{j=0}^{\infty} \frac{(\sqrt{z\zeta})^{2j+k}}{j!\Gamma(j+k+1)} \left\{ c_{j+n,\, j+k+q} \left(\ln \sqrt{z\zeta} - \frac{\psi(j+1)+\psi(j+k+1)}{2} \right) + \right.$$

$$\left. + \frac{1}{2} (c^{*}_{j+n-1,\, j+k+q} + c^{*}_{j+k+q-1,\, j+n}) \right\} \qquad (k = 0, 1, 2, \ldots)$$

Therefore, (4.23), (4.24) and (4.27) define expansions of the functions (4.22) in polar coordinates, i.e. the solutions $T_i^{(-n)}(z, \zeta)$ and their different derivatives.

5. Now, let us examine representations of solutions of the boundary value problems for a multiply connected domain. If B is a finite multiply connected domain whose complement $B_1, B_2, \ldots, B_m$ are bounded continua, and B_0 contains the infinite point, then the solution $F(z, \zeta)$ can be sought in the form

$$F(z,\zeta) = \sum_{n=0}^{\infty} \{A_n \Phi_n (z - z_0, \zeta - \zeta_0) + A^{*}_n \Phi^{*}_n (z - z_0, \zeta - \zeta_0) +$$

$$+ B_n \Psi_n (z - z_0, \zeta - \zeta_0) + B_n^{*} \Psi_n^{*} (z - z_0, \zeta - \zeta_0)\} + \sum_{j=1}^{m} \sum_{n=0}^{\infty} \times \tag{5.1}$$

$$\times \{A_{1,n}^{(j)} T_1^{(-n)} (z - z_j, \zeta - \zeta_j) + A_{2,n}^{(j)} T_2^{(-n)} (z - z_j, \zeta - \zeta_j) + A_{3,n}^{(j)} T_3^{(-n)} (z - z_j,$$

$$\zeta - \zeta_j) + A_{4,n}^{(j)} T_4^{(-n)} (z - z_j, \zeta - \zeta_j)\}$$

where the points $(z_0, \zeta_0) \in B_0$, $(z_j, \zeta_j) \in B_j$, $(j = 1, 2, \ldots, m)$. Constants in (5.1) are defined from the boundary conditions of the appropriate boundary value problem.

If B is an unbounded multiply connected domain containing the infinite point, then the first sum in (5.1) vanishes, i.e.

$$A_n = A_n^{*} = B_n = B_n^{*} = 0$$

In the case of the periodic problem (an unbounded shell with a periodic system of holes), the representation (5.1) simplifies somewhat.

We have

$$F(z,\zeta) = \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \{A_{1,j}^{(n)} T_1^{(-j)} (z - z_n, \zeta - \zeta_n) + A_{2,j}^{(n)} T_2^{(-j)} (z - z_n, \zeta - \zeta_n) +$$

$$+ A_{3,j}^{(n)} T_3^{(-j)} (z - z_n, \zeta - \zeta_n) + A_{4,j}^{(n)} T_4^{(-j)} (z - z_n, \zeta - \zeta_n)\} \tag{5.2}$$

where $z_n = z_0 + n\omega$, and ω is the fundamental period.

We find
$$A_{k,j}^{(n)} = A_{k,j} \qquad (k = 1, 2, 3, 4) \tag{5.3}$$
from the condition of periodicity of the stresses under the assumption of a possibility of changing the order of summation.

Introducing the periodic functions (the series on the right side always converges since the functions $T_i^{(-j)}$ decay exponentially at infinity)
$$\Pi_i^{(-j)}(z, \zeta) = \sum_{n=-\infty}^{\infty} T_i^{(-j)}(z - z_n, \zeta - \zeta_n) \tag{5.4}$$
we obtain the final representation
$$F(z, \zeta) = \sum_{j=0}^{\infty} \{A_{1,j}\Pi_1^{(-j)}(z, \zeta) + A_{2,j}\Pi_2^{(-j)}(z, \zeta) + A_{3,j}\Pi_3^{(-j)}(z, \zeta) + A_{4,j}\Pi_4^{(-j)}(z, \zeta)\} \tag{5.5}$$

We analogously obtain a representation of the solution in the case of a doubly-periodic problem
$$F(z, \zeta) = \sum_{j=0}^{\infty} \{A_{1,j}m_1^{(-j)}(z, \zeta) + A_{2,j}m_2^{(-j)}(z, \zeta) + A_{3,j}m_3^{(-j)}(z, \zeta) + A_{4,j}m_4^{(-j)}(z, \zeta)\} \tag{5.6}$$
where the doubly-periodic functions $m_i^{(-j)}(z, \zeta)$ are
$$m_i^{(-j)}(z, \zeta) = \sum_{m,n} T_i^{(-j)}(z - z_{m,n}, \zeta - \zeta_{m,n}) \qquad (i = 1, 2, 3, 4; \quad j = 0, 1, 2, \ldots)$$

6. Let us examine the case of a simply-connected unbounded domain (an unbounded shell with a hole) in more detail. In this case we evidently have
$$F(z, \zeta) = \sum_{n=0}^{\infty} \{A_{1,n}T_1^{(-n)}(z, \zeta) + A_{2,n}T_2^{(-n)}(z, \zeta) + A_{3,n}T_3^{(-n)}(z, \zeta) + A_{4,n}T_4^{(-n)}(z, \zeta)\} \tag{6.1}$$

According to equalities (4.22), (4.23), the expression (6.1) can be represented in polar coordinates as follows:
$$\frac{\partial^{p+q}}{\partial z^p \partial \zeta^q} F(z, \zeta) = \sum_{k=-\infty}^{\infty} F_{p,q}^k(\sqrt{z\zeta})\, e^{ik\theta} \tag{6.2}$$
$$F_{p,q}^k = \sum_{n=0}^{\infty} \{[A_{1,n} + (-1)^{k+p+q} A_{3,n}]\, F_{p,q}^{k,n}(\sqrt{z\zeta}) + [A_{2,n} + (-1)^{k+p+q} A_{4,n}] \times$$
$$\times F_{q,p}^{-k,n}(\sqrt{z\zeta})\}$$

The quantities $F_{p,q}^{k,n}$ are given in (4.27).

If the domain and the loading are symmetric with respect to the coordinate axes, then $A_{1,n} = A_{2,n} = A_{3,n} = A_{4,n} = A_n$ from the symmetry conditions, and the representation (6.1) simplifies. We have
$$F(z, \zeta) = \sum_{n=0}^{\infty} A_n T^{(-n)}(z, \zeta) \tag{6.3}$$
$$T^{(-n)}(z, \zeta) = T_1^{(-n)}(z, \zeta) + T_1^{(-n)}(\zeta, z) + T_1^{(-n)}(-z, -\zeta) + T_1^{(-n)}(-\zeta, -z)$$

We have the representation (6.2) in polar coordinates, where

$$F_{p,q}^{k}(\sqrt{z\zeta}) = [1 + (-1)^{k+q+p}] \sum_{n=0}^{\infty} (F_{p,q}^{k,n} + F_{q,p}^{-k,n}) A_n \tag{6.4}$$

For example, for the solution $F(z, \zeta)$ we have the representation

$$F(z, \zeta) = F_{0,0}(\sqrt{z\zeta}) + 2\sum_{k=1}^{\infty} F_{0,0}^{2k}(\sqrt{z\zeta}) \cos 2k\theta \tag{6.5}$$

If the problem is inversely symmetric relative to the x- and y-axes, then

$$A_{1,n} = A_{3,n}, \quad A_{2,n} = A_{4,n}, \qquad A_{1,n} = -A_{2,n} = A_n \tag{6.6}$$

We obtain

$$F(z, \zeta) = \sum_{n=0}^{\infty} A_n T^{(-n)}(z, \zeta) \tag{6.7}$$

$$T^{(-n)}(z, \zeta) = T_1^{(-n)}(z, \zeta) - T_1^{(-n)}(\zeta, z) + T_1^{(-n)}(-z, -\zeta) - T_1^{(-n)}(-\zeta, -z)$$

The expansion in polar coordinates remains valid, and its coefficients are

$$F_{p,q}^{k}(\sqrt{z\zeta}) = [1 + (-1)^{k+p+q}] \sum_{n=0}^{\infty} A_n (F_{p,q}^{k,n} - F_{q,p}^{-k,n}) \tag{6.8}$$

For example, the solution $F(z, \zeta)$ is represented by the formula

$$F(z, \zeta) = 2i \sum_{k=1}^{\infty} F_{0,0}^{2k} \sin 2k\theta \tag{6.9}$$

7. Only conditions for the uniqueness of the tangential displacements require special consideration.

If we have the representations (6.1) in mind, then the uniqueness condition yields the relationship

$$\sum_{n=0}^{\infty} (A_{1,n} - A_{3,n}) = i \sum_{n=0}^{\infty} (\overline{A}_{2,n} - \overline{A}_{4,n}) \tag{7.1}$$

In particular, it follows from (7.1) that the uniqueness conditions of the tangential displacements are satisfied automatically for the symmetric $(A_{1,n} = A_{2,n} = A_{3,n} = A_{4,n})$ and inversely symmetric $(A_{1,n} = A_{3,n} = -A_{2,n} = -A_{4,n})$ problems.

The above-mentioned representations and their expressions in polar coordinates can be utilized directly to solve in series various boundary value problems of shallow shell theory.

BIBLIOGRAPHY

1. Grigoliuk, E. I. and Fil'shtinskii, L. A., Perforated plates and shells and associated problems. Survey of Results. In the book: "Elasticity and Plasticity", 1965. Surveys of Science, Mechanics Series, Moscow, VINITI, 1967.
2. Fil'shtinskii, L. A., Complete systems of particular solutions in shallow shell theory. PMM Vol. 33, №4, 1969.
3. Sneddon, I. N. and Berry, D. C., Classical Elasticity Theory (Russian translation). Moscow, Fizmatgiz, 1961.

4. Muskhelishvili, N. I., Some Fundamental Problems of Mathematical Elasticity Theory. 4th Ed., Moscow, Akad. Nauk SSSR Press, 1954.
5. Bateman, H. and Erdelyi, A., Higher Transcendental Functions (Russian transalation), Moscow, "Nauka" Press, 1966.
6. Gol'denveizer, A. L., Theory of Elastic Thin Shells, Moscow, Gostekhizdat, 1953.

Translated by M. D. F.

ON THE OPTIMAL DISTRIBUTION OF THE RESISTIVITY TENSOR OF THE WORKING SUBSTANCE IN A MAGNETOHYDRODYNAMIC CHANNEL

PMM Vol. 34, №2, 1970, pp. 270-291
K. A. LUR'E
(Leningrad)
(Received September 15, 1969)

The analysis of the problem formulated and studied in [1, 2] is resumed. The conditions which the distribution of the resistivity of the working substance in a channel with finite electrodes must satisfy in order for the current in the external circuit to reach its maximum are investigated. The resistivity is assumed to be a tensor function of the coordinates; the tensor is assumed to be symmetric and its principal values to be piecewise-continuously differentiable functions.

1. Formulation of the problem. We consider a flat channel (Fig. 1) of width 2δ whose walls are dielectric everywhere except for two segments of equal length 2λ facing each other at opposite sides of the channel; these segments are made of an ideally conductive material. The conductive segments are connected through the load R.

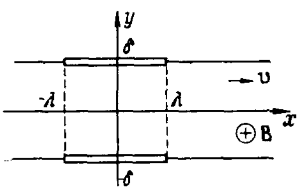

Fig. 1

The working substance characterized by the resistivity tensor $P_0(x, y)$ which varies from point to point is moving in the channel at the velocity $v\,(V(y), 0, 0)$. We assume that this center is symmetric; let $\rho_1(x, y)$, $\rho_2(x, y)$ be its principal values and α, β the corresponding principal axes. Denoting the angle between the positive direction of the x-axis and the α-axis (*) by $\gamma(x, y)$, we can find the Cartesian components of the tensor P_0 from the formulas

$$\rho_{xx} = {}^1/_2\,[\rho_1 + \rho_2 + (\rho_1 - \rho_2)\cos 2\gamma], \quad \rho_{yy} = {}^1/_2\,[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\gamma] \tag{1.1}$$

$$\rho_{xy} = \rho_{yx} = {}^1/_2\,(\rho_1 - \rho_2)\sin 2\gamma$$

Imposition of a magnetic field $\mathbf{B}\,(0, 0, -B(x))$ causes an electric current of density $\mathbf{j}$ to flow in the channel (the Cartesian coordinates of this vector will be denoted

*) We assume that the α- and β-axes form a right-handed system.